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# Robust Optimal $H_\infty$ Control for Uncertain 2-D Discrete State-delayed Systems described by the General Model via Memory State Feedback

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**Abstract—***This paper considers with the problem of robust optimal $H_\infty$ control for uncertain two-dimensional (2-D) discrete state-delayed systems described by the general model (GM) via memory state feedback. The parameter uncertainties are assumed to be unknown but norm-bounded. A certain linear matrix inequality (LMI)-based sufficient condition is established for the existence of $\gamma$ -suboptimal robust $H_\infty$ memory state feedback controller. Moreover, a convex optimization problem is developed to design a robust optimal $H_\infty$ memory state feedback controller which minimizes the value of $H_\infty$ noise attenuation level $\gamma$ of the resulting closed-loop system. Finally, an illustrative example is given to demonstrate the effectiveness of proposed method and a comparison of our result with state feedback without memory is also made.*

**Keywords** 2-*D discrete systems; $H_\infty$ control; linear matrix inequality; memory state feedback; general model*

## 1. INTRODUCTION

In recent years, several control design techniques have been studied in literature for the disturbance attenuation problem, such as $H_\infty$ control, $l_2 - l_\infty$ control, $H_2/H_\infty$ control, passivity control and so on (Mathiyalagan et al., 2015). Among them, $H_\infty$ control technique has drawn great attention in the research community due to their wide application and usages. A major advantage of $H_\infty$ control is that its performance specification takes account of the worst-case performance for system in terms of system energy gain (Xu and Yu, 2008).

For synthesis problem of the uncertain 2-D discrete state-delayed system, state feedback controller is of two types (Azuma, et al. 2002; Xu et al., 2002; Yang et al., 2010): the memoryless state feedback controller and the memory state feedback controller. It is mentioned in (Azuma et al., 2002; Moon et al., 2001; Yang et al., 2010; Zames, 1981) that when the size of time-delay information can be obtained, the memory state feedback controllers have improved performance and are less conservative than the memoryless one which motivates the present study.

Therefore, in this paper, we have proposed the problem of robust optimal $H_\infty$ control for uncertain 2-D discrete state-delayed systems described by GM via memory state feedback. The parameter uncertainty is assumed to be unknown but norm-bounded. The approach adopted in this paper is as follows: we first develop a sufficient condition for the existence of $\gamma$-suboptimal robust $H_\infty$ memory state feedback controller in terms of a certain linear matrix inequality (LMI). Further, a convex optimization problem is introduced to select a robust optimal $H_\infty$ memory state feedback controller, which minimizes the value of $H_\infty$ noise attenuation level $\gamma$ of the resulting closed-loop system. Finally, an example is given to illustrate the effectiveness of the proposed approach.

## 2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the uncertain 2-D discrete state-delayed systems described by the GM [Kurek, 1985].

$$\boldsymbol{x}(i+1, j+1) = \overline{\boldsymbol{A}}_1 \boldsymbol{x}(i, j+1) + \overline{\boldsymbol{A}}_2 \boldsymbol{x}(i+1, j) + \overline{\boldsymbol{A}}_0 \boldsymbol{x}(i, j)$$

$+\bar{A}_{1d}x(i-d_1,j+1)+\bar{A}_{2d}x(i+1,j-d_2)$

$+\bar{A}_{0d}x(i-k_1,j-k_2)+\bar{B}_1w(i,j+1)$

$+\bar{B}_2w(i+1,j)+\bar{B}_0w(i,j)+\bar{C}_1u(i,j+1)$

$+\bar{C}_2u(i+1,j)+\bar{C}_0u(i,j)$, (2.1a)

$z(i,j)=Hx(i,j)+Lw(i,j)$, (2.1b)

where $x(i,j)\in R^n$ is the state vector, $u(i,j)\in R^m$ is the input vector, $z(i,j)\in R^p$ is the controlled output, $w(i,j)\in R^q$ is the noise input which belongs to $\ell_2\{[0,\infty),[0,\infty)\}$ and

$\bar{A}_1=(A_1+\Delta A_1)$, $\bar{A}_2=(A_2+\Delta A_2)$, $\bar{A}_0=(A_0+\Delta A_0)$, $\bar{A}_{1d}=(A_{1d}+\Delta A_{1d})$, $\bar{A}_{2d}=(A_{2d}+\Delta A_{2d})$, $\bar{A}_{0d}=(A_{0d}+\Delta A_{0d})$, $\bar{B}_1=(B_1+\Delta B_1)$, $\bar{B}_2=(B_2+\Delta B_2)$, $\bar{B}_0=(B_0+\Delta B_0)$, $\bar{C}_1=(C_1+\Delta C_1)$, $\bar{C}_2=(C_2+\Delta C_2)$, $\bar{C}_0=(C_0+\Delta C_0)$.

(2.1c)

The matrices $A_1, A_2, A_0, A_{1d}, A_{2d}, A_{0d}\in R^{n\times n}$, $B_1, B_2, B_0\in R^{n\times q}$, $C_1, C_2, C_0\in R^{n\times m}$, $H\in R^{p\times n}$ , and $L\in R^{p\times q}$ are known constant matrices representing the nominal plant. The matrices $\Delta A_1,\Delta A_2,\Delta A_0$, $\Delta B_1,\Delta B_2,\Delta B_0$, $\Delta C_1,\Delta C_2$, and $\Delta C_0$ represent parameter uncertainties in the system matrices, which are assumed to be of the form

$\begin{bmatrix}\Delta A_1 & \Delta A_2 & \Delta A_0 & \Delta A_{1d} & \Delta A_{2d} & \Delta A_{0d}\end{bmatrix},=H_0F(i,j)\begin{bmatrix}E_1 & E_2 & E_3 & E_{1d} & E_{2d} & E_{0d}\end{bmatrix}$

$\begin{bmatrix}\Delta B_1 & \Delta B_2 & \Delta B_0\end{bmatrix}=H_0F(i,j)\begin{bmatrix}E_4 & E_5 & E_6\end{bmatrix}$,

$\begin{bmatrix}\Delta C_1 & \Delta C_2 & \Delta C_0\end{bmatrix}=H_0F(i,j)\begin{bmatrix}E_7 & E_8 & E_9\end{bmatrix}$,

(2.1d)

where $H_0\in R^{n\times k}$, $E_1, E_2, E_3, E_{1d}$, $E_{2d}, E_{0d}\in R^{l\times n}$, $E_4, E_5, E_6\in R^{l\times q}$ and $E_7, E_8, E_9\in R^{l\times m}$ are known structural matrices of uncertainty. $F(i,j)\in R^{k\times l}$ is an unknown matrix representing parameter uncertainty and satisfies

$F^T(i,j)F(i,j)\le I$ (or equivalently,

$\|F(i,j)\|\le 1$). (2.2)

The following well known lemmas are needed in the proof of the main results.

**Lemma 1** (Du et al., 2001; Guan et al., 2001; Xie and Soh, 1995) Let $A\in R^{n\times n}$, $H_0\in R^{n\times k}$, $E\in R^{l\times n}$ and $Q=Q^T\in R^{n\times n}$ be given matrices. Then there exists a positive definite matrix $P$ such that

$[A+H_0FE]^TP[A+H_0FE]-Q<0$ (2.4)

for all $F$ satisfying $\|F(i,j)\|\le I$, if and only if there exists a scalar $\varepsilon>0$ such that

$$\begin{bmatrix}-P^{-1}+\varepsilon H_0H_0^T & A\\ A^T & \varepsilon^{-1}E^TE-Q\end{bmatrix}<\mathbf{0}. \quad (2.5)$$

**Lemma 2** (Boyd et al., 1994) For real matrices $M, L, Q$ of appropriate dimension, where $M=M^T$ and $Q=Q^T>\mathbf{0}$ then $M+L^TQL<\mathbf{0}$, if and only if

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{L}^T \\ \boldsymbol{L} & \boldsymbol{-Q}^{-1} \end{bmatrix} < \boldsymbol{0} \qquad (2.6)$$

or equivalently

$$\begin{bmatrix} \boldsymbol{-Q}^{-1} & \boldsymbol{L} \\ \boldsymbol{L}^T & \boldsymbol{M} \end{bmatrix} < \boldsymbol{0}. \qquad (2.7)$$

**Theorem 1.** Consider the system (2.1) with $\boldsymbol{u}(i,j)=\boldsymbol{0}$ and initial condition (2.3), for a given positive scalar $\gamma$, if there exist symmetric positive definite matrices $\boldsymbol{P}, \boldsymbol{P}_1, \boldsymbol{P}_2 \in R^{n\times n}$, $\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3 \in R^{n\times n}$ satisfying $\boldsymbol{P}_1 > \gamma^2 \boldsymbol{Q}_1$, $\boldsymbol{P}_2 > \gamma^2 \boldsymbol{Q}_2$, $\boldsymbol{0} < \boldsymbol{P} - \boldsymbol{P}_1 - \boldsymbol{P}_2 < \gamma^2 \boldsymbol{Q}_3$, $\boldsymbol{R}_1 < \gamma^2 \boldsymbol{Z}_1$, $\boldsymbol{R}_2 < \gamma^2 \boldsymbol{Z}_2$, and $\boldsymbol{R}_3 < \gamma^2 \boldsymbol{Z}_3$, such that the following matrix inequality

$$\left( \begin{bmatrix} \overline{\boldsymbol{A}}_1^T \\ \overline{\boldsymbol{A}}_2^T \\ \overline{\boldsymbol{A}}_0^T \\ \overline{\boldsymbol{A}}_{1d}^T \\ \overline{\boldsymbol{A}}_{2d}^T \\ \overline{\boldsymbol{A}}_{0d}^T \\ \overline{\boldsymbol{B}}_1^T \\ \overline{\boldsymbol{B}}_2^T \\ \overline{\boldsymbol{B}}_0^T \end{bmatrix} \boldsymbol{P} \begin{bmatrix} \overline{\boldsymbol{A}}_1^T \\ \overline{\boldsymbol{A}}_2^T \\ \overline{\boldsymbol{A}}_0^T \\ \overline{\boldsymbol{A}}_{1d}^T \\ \overline{\boldsymbol{A}}_{2d}^T \\ \overline{\boldsymbol{A}}_{0d}^T \\ \overline{\boldsymbol{B}}_1^T \\ \overline{\boldsymbol{B}}_2^T \\ \overline{\boldsymbol{B}}_0^T \end{bmatrix}^T + \right.$$

$$\left. \begin{bmatrix} \boldsymbol{-P}_1 + \boldsymbol{R}_1 + \boldsymbol{H}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}^T\boldsymbol{L} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{-P}_2 + \boldsymbol{R}_2 + \boldsymbol{H}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}^T\boldsymbol{L} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{-P} + \boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{R}_3 + \boldsymbol{H}^T\boldsymbol{H}) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}^T\boldsymbol{L} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{R}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{R}_2 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{R}_3 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{L}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{L} - \gamma^2\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{L} - \gamma^2\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{L} - \gamma^2\boldsymbol{I} \end{bmatrix} \right) < \boldsymbol{0} \qquad (2.9)$$

holds, then the system (2.1) is asymptotically stable and preserves $H_\infty$ noise attenuation $\gamma$ against external disturbances.

## 3. $H_\infty$ CONTROLLER DESIGN

Consider that the system states are available for feedback, then our aim is to design the following memory state feedback control law

$$\boldsymbol{u}(i,j+1)=\boldsymbol{K}_1\boldsymbol{x}(i,j+1)+\boldsymbol{K}_2\,\boldsymbol{x}(i-d_1,j+1), \tag{3.1a}$$

$$\boldsymbol{u}(i+1,j)=\boldsymbol{K}_3\boldsymbol{x}(i+1,j)+\boldsymbol{K}_4\boldsymbol{x}(i+1,j-d_2), \tag{3.1b}$$

$$\boldsymbol{u}(i,j)=\boldsymbol{K}_5\boldsymbol{x}(i,j)+\boldsymbol{K}_6\boldsymbol{x}(i-k_1,j-k_2), \tag{3.1c}$$

where $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, $\boldsymbol{K}_3$, $\boldsymbol{K}_4$, $\boldsymbol{K}_5$ and $\boldsymbol{K}_6$ are appropriately dimensioned stabilizing control law matrices, such that the resulting closed-loop system is given by

$$\begin{aligned}\boldsymbol{x}\left(i+1,j+1\right)=&(\bar{\boldsymbol{A}}_1+\bar{\boldsymbol{C}}_1\boldsymbol{K}_1)\,\boldsymbol{x}(i,j+1)\\&+(\bar{\boldsymbol{A}}_2+\bar{\boldsymbol{C}}_2\boldsymbol{K}_3)\,\boldsymbol{x}(i+1,j)\\&+(\bar{\boldsymbol{A}}_0+\bar{\boldsymbol{C}}_0\boldsymbol{K}_5)\,\boldsymbol{x}(i,j)+(\bar{\boldsymbol{A}}_{1d}\\&+\bar{\boldsymbol{C}}_1\boldsymbol{K}_2)\,\boldsymbol{x}(i\text{ - }d_1,j+1)\\&+(\bar{\boldsymbol{A}}_{2d}+\bar{\boldsymbol{C}}_2\boldsymbol{K}_4)\,\boldsymbol{x}(i+1,j\text{ - }d_2)\\&+(\bar{\boldsymbol{A}}_{0d}+\bar{\boldsymbol{C}}_0\boldsymbol{K}_6)\,\boldsymbol{x}(i\text{ - }k_1,j\text{ - }k_2)\\&+\bar{\boldsymbol{B}}_2\boldsymbol{w}(i+1,j)+\bar{\boldsymbol{B}}_0\boldsymbol{w}(i,j),\\&+\bar{\boldsymbol{B}}_1\boldsymbol{w}(i,j+1)\,,\end{aligned} \tag{3.2a}$$

$$\boldsymbol{z}(i,j)=\boldsymbol{H}\boldsymbol{x}(i,j)+\boldsymbol{L}\boldsymbol{w}(i,j)\,. \tag{3.2b}$$

**Theorem 2** Consider the system (2.1) with initial condition (2.3). Given two scalars $\gamma>0$ and $\varepsilon>0,$ if there exist a matrix $\boldsymbol{N}\in R^{m\times n}$ and symmetric positive definite matrices $\bar{\boldsymbol{P}},\bar{\boldsymbol{P}}_1,\bar{\boldsymbol{P}}_2,\bar{\boldsymbol{R}}_1,\bar{\boldsymbol{R}}_2,\bar{\boldsymbol{R}}_3\in R^{n\times n}$ such that

$$\left[\begin{array}{ccccccccccc}
-\bar{\boldsymbol{P}}_1+\bar{\boldsymbol{R}}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\boldsymbol{P}}\boldsymbol{A}_1^T+\boldsymbol{N}_1^T\boldsymbol{C}_1^T & \bar{\boldsymbol{P}}\boldsymbol{H}^T\\
* & -\bar{\boldsymbol{P}}_2+\bar{\boldsymbol{R}}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\boldsymbol{P}}\boldsymbol{A}_2^T+\boldsymbol{N}_3^T\boldsymbol{C}_2^T & \mathbf{0}\\
* & * & \begin{array}{c}(-\bar{\boldsymbol{P}}+\bar{\boldsymbol{P}}_1\\+\bar{\boldsymbol{P}}_2+\bar{\boldsymbol{R}}_3)\end{array} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\boldsymbol{P}}\boldsymbol{A}_0^T+\boldsymbol{N}_5^T\boldsymbol{C}_0^T & \mathbf{0}\\
* & * & * & -\bar{\boldsymbol{R}}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\boldsymbol{P}}\boldsymbol{A}_{1d}^T+\boldsymbol{N}_2^T\boldsymbol{C}_1^T & \mathbf{0}\\
* & * & * & * & -\bar{\boldsymbol{R}}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\boldsymbol{P}}\boldsymbol{A}_{2d}^T+\boldsymbol{N}_4^T\boldsymbol{C}_2^T & \mathbf{0}\\
* & * & * & * & * & -\bar{\boldsymbol{R}}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\boldsymbol{P}}\boldsymbol{A}_{0d}^T+\boldsymbol{N}_6^T\boldsymbol{C}_0^T & \mathbf{0}\\
* & * & * & * & * & * & -\gamma^2\boldsymbol{I} & \mathbf{0} & \mathbf{0} & \boldsymbol{B}_1^T & \boldsymbol{L}^T\\
* & * & * & * & * & * & * & -\gamma^2\boldsymbol{I} & \mathbf{0} & \boldsymbol{B}_2^T & \mathbf{0}\\
* & * & * & * & * & * & * & * & -\gamma^2\boldsymbol{I} & \boldsymbol{B}_0^T & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & -\bar{\boldsymbol{P}}+\varepsilon\boldsymbol{H}_0\boldsymbol{H}_0^T & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & * & -\boldsymbol{I}\\
* & * & * & * & * & * & * & * & * & * & *\\
* & * & * & * & * & * & * & * & * & * & *\\
* & * & * & * & * & * & * & * & * & * & *\\
* & * & * & * & * & * & * & * & * & * & *\\
* & * & * & * & * & * & * & * & * & * & *\\
* & * & * & * & * & * & * & * & * & * & *\\
* & * & * & * & * & * & * & * & * & * & *\\
* & * & * & * & * & * & * & * & * & * & *
\end{array}\right.$$

$$\left.\begin{matrix}
\mathbf{0} & \mathbf{0} & \bar{P}E_1^T & E_7N_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\bar{P}H^T & \mathbf{0} & \bar{P}E_2^T & \mathbf{0} & E_8N_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \bar{P}H^T & \bar{P}E_3^T & \mathbf{0} & \mathbf{0} & E_9N_5 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \bar{P}E_{1d}^T & E_7N_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \bar{P}E_{2d}^T & \mathbf{0} & E_8N_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \bar{P}E_{0d}^T & \mathbf{0} & \mathbf{0} & E_9N_6 & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & E_4^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
L^T & \mathbf{0} & E_5^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & L^T & E_6^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & -I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & -\varepsilon I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & -\varepsilon I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & -\varepsilon I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & -\varepsilon I & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & -\varepsilon I & \mathbf{0} \\
* & * & * & * & * & * & * & -\varepsilon I
\end{matrix}\right] < \mathbf{0}, \tag{3.3}$$

then the closed-loop system (3.2) has a specified $H_\infty$ noise attenuation $\gamma$ and the stabilizing control law matrices with

$$K_n = N_n \bar{P}^{-1}, \text{ (where n=1 to 6)} \quad (3.4a)$$

is $\gamma$ -suboptimal robust $H_\infty$ memory state feedback controllers for the system (2.1).

**Proof**: Extending the matrix inequality (2.9) for the closed-loop system (3.2), we

$$\left(\begin{bmatrix} (\bar{A}_1+\bar{C}_1K_1)^T \\ (\bar{A}_2+\bar{C}_2K_3)^T \\ (\bar{A}_0+\bar{C}_0K_5)^T \\ (\bar{A}_{1d}+\bar{C}_1K_2)^T \\ (\bar{A}_{2d}+\bar{C}_2K_4)^T \\ (\bar{A}_{0d}+\bar{C}_0K_6)^T \\ \bar{B}_1^T \\ \bar{B}_2^T \\ \bar{B}_0^T \end{bmatrix} P \begin{bmatrix} (\bar{A}_1+\bar{C}_1K_1)^T \\ (\bar{A}_2+\bar{C}_2K_3)^T \\ (\bar{A}_0+\bar{C}_0K_5)^T \\ (\bar{A}_{1d}+\bar{C}_1K_2)^T \\ (\bar{A}_{2d}+\bar{C}_2K_4)^T \\ (\bar{A}_{0d}+\bar{C}_0K_6)^T \\ \bar{B}_1^T \\ \bar{B}_2^T \\ \bar{B}_0^T \end{bmatrix}^T + \right.$$

$$\left[\begin{matrix}
-P_1+R_1+H^TH & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -P_2+R_2+H^TH & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & (-P+P_1+P_2 +R_3+H^TH) \\
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
L^TH & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & L^TH & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & L^TH
\end{matrix}\right.$$

$$\left.\left.\begin{matrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{H}^T\boldsymbol{L} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{H}^T\boldsymbol{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{H}^T\boldsymbol{L} \\ -\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{R}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{R}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{L}^T\boldsymbol{L}-\gamma^2\boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{L}^T\boldsymbol{L}-\gamma^2\boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{L}^T\boldsymbol{L}-\gamma^2\boldsymbol{I} \end{matrix}\right]\right) < \mathbf{0}. \quad (3.5)$$

Applying Lemma 1 and 2 to the matrix inequality (3.5), then pre-multiplying and post-multiplying both sides of the inequality (3.7) by $diag\left\{\boldsymbol{P}^{-1},\boldsymbol{P}^{-1},\boldsymbol{P}^{-1},\boldsymbol{P}^{-1},\boldsymbol{P}^{-1},\boldsymbol{P}^{-1},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I}\right\}$, and denoting $\bar{\boldsymbol{P}}=\boldsymbol{P}^{-1}$, $\bar{\boldsymbol{P}}_1=\bar{\boldsymbol{P}}\boldsymbol{P}_1\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{P}}_2=\bar{\boldsymbol{P}}\boldsymbol{P}_2\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{R}}_1=\bar{\boldsymbol{P}}\boldsymbol{R}_1\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{R}}_2=\bar{\boldsymbol{P}}\boldsymbol{R}_2\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{R}}_3=\bar{\boldsymbol{P}}\boldsymbol{R}_3\bar{\boldsymbol{P}}$, and $\boldsymbol{N}_1=\boldsymbol{K}_1\bar{\boldsymbol{P}}$, $\boldsymbol{N}_2=\boldsymbol{K}_2\bar{\boldsymbol{P}}$, $\boldsymbol{N}_3=\boldsymbol{K}_3\bar{\boldsymbol{P}}$, $\boldsymbol{N}_4=\boldsymbol{K}_4\bar{\boldsymbol{P}}$, $\boldsymbol{N}_5=\boldsymbol{K}_5\bar{\boldsymbol{P}}$, $\boldsymbol{N}_6=\boldsymbol{K}_6\bar{\boldsymbol{P}}$, we obtain that the equivalence of (3.7) and (3.3) follows trivially by applying lemma 2. This completes the proof of theorem 2.

Based on Theorem 2, the design problem of a robust optimal $H_\infty$ controller can be formulated as:

$$\text{minimize}\ \gamma^2 \qquad (3.8)$$

s. t. (3.3).

## 4. AN ILLUSTRATIVE EXAMPLE

*Example* 4.1: In this example, a comparison of our result with previously reported result *i.e.* state feedback without memory (Singh and A. Dhawan, 2016) is made. Consider an uncertain 2-D discrete state-delayed system described by (2.1) and initial condition (2.3) with

$$\boldsymbol{A}_1=\begin{bmatrix} .60 & 1 \\ 0.02 & 0 \end{bmatrix},\ \boldsymbol{A}_2=\begin{bmatrix} 0 & 0 \\ 0.1 & 0.6 \end{bmatrix},\quad \boldsymbol{A}_0=\begin{bmatrix} 0 & 0 \\ 0 & 0.21 \end{bmatrix},\ \boldsymbol{A}_{1d}=\begin{bmatrix} 0 & 0 \\ 0 & 0.03 \end{bmatrix},\ \boldsymbol{A}_{2d}=\begin{bmatrix} 0 & 0 \\ 0 & 0.09 \end{bmatrix},\ \boldsymbol{A}_{0d}=\begin{bmatrix} 0 & 0 \\ 0 & 0.02 \end{bmatrix},\ \boldsymbol{C}_1=\begin{bmatrix} 0 \\ 0.002 \end{bmatrix},\ \boldsymbol{C}_2=\begin{bmatrix} 0 \\ 0.04 \end{bmatrix},$$

$$\boldsymbol{C}_0=\begin{bmatrix} 0 \\ 0.01 \end{bmatrix}. \qquad (4.1)$$

By considering, the problem of $H_\infty$ disturbance attenuation with

$$\boldsymbol{B}_1=\begin{bmatrix} 0 \\ 0.04 \end{bmatrix},\quad \boldsymbol{B}_2=\begin{bmatrix} 0 \\ .003 \end{bmatrix},\quad \boldsymbol{B}_0=\begin{bmatrix} 0 \\ .02 \end{bmatrix},\quad \boldsymbol{H}=\begin{bmatrix} 0.01 & 0.01 \end{bmatrix},\quad \boldsymbol{L}=0.5. \qquad (4.2)$$

It is also assumed that the above system is subjected to the parameter uncertainties of the form (2.1c) and (2.1d) with

$$\boldsymbol{H}_0=\begin{bmatrix} 0.001 & 0.002 \\ 0 & 0 \end{bmatrix},\ \boldsymbol{E}_1=\begin{bmatrix} 0.03 & 0 \\ 0 & 0.03 \end{bmatrix},\quad \boldsymbol{E}_2=\begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix},\quad \boldsymbol{E}_3=\begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix},\ \boldsymbol{E}_{1d}=\begin{bmatrix} 0.09 & 0 \\ 0 & 0.09 \end{bmatrix},$$

$$\boldsymbol{E}_{2d}=\begin{bmatrix} 0.06 & 0 \\ 0 & 0.06 \end{bmatrix},\ \boldsymbol{E}_{0d}=\begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix},\ \boldsymbol{E}_4=\begin{bmatrix} 0.1 \\ 0 \end{bmatrix},\ \boldsymbol{E}_5=\begin{bmatrix} -0.007 \\ 0 \end{bmatrix},\quad \boldsymbol{E}_6=\begin{bmatrix} -0.7 \\ 0 \end{bmatrix},\ \boldsymbol{E}_7=\begin{bmatrix} -0.001 \\ 0 \end{bmatrix},\ \boldsymbol{E}_8=\begin{bmatrix} 0.2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{E}_9=\begin{bmatrix} 0.3 \\ 0 \end{bmatrix},\ d_1=5,\ d_2=3,\ k_1=3,\ k_2=2.$$

(4.3)

Using Matlab LMI tool box (Boyd et al., 1994), it is found that the optimization problem (3.8) is feasible for present example and the robust optimal $H_\infty$ memory state feedback controllers and optimal $H_\infty$ noise attenuation level are given as

$$\boldsymbol{K}_1 = [\text{-329.2774} \quad \text{-488.9344}],\ \boldsymbol{K}_2 = [\text{-0.0035} \quad \text{-15.0003}],\ \boldsymbol{K}_3 = [\text{-4.8779} \quad \text{-17.0267}],\ \boldsymbol{K}_4 = [\text{-0.0007} \quad \text{-2.0582}],\ \boldsymbol{K}_5 = [\text{-9.5729} \quad \text{-15.7668}],$$
$$\boldsymbol{K}_6 = [\text{-0.1285} \quad \text{-0.9719}],\quad \gamma^* = 0.5015. \tag{4.4}$$

For validation of result, Figure 1 shows the frequency response for the system (3.2) over all frequencies, *i.e*., $\left|\boldsymbol{G}(e^{j\omega_1}, e^{j\omega_2})\right|$, $0 \le \omega_1 \le 2\pi$, $0 \le \omega_2 \le 2\pi$. It is apparent from Figure 1 that the maximum value of $\left|\boldsymbol{G}(e^{j\omega_1}, e^{j\omega_2})\right|$ is 0.5005, which is below the specified level of attenuation $\gamma^* = 0.5015$.

For state feedback without memory, *i.e*. example 1 of (Singh and Dhawan, 2016), the optimal solution is obtained as

$$\boldsymbol{N} = [115.1984 \quad \text{-143.0584}],\ \gamma = 0.5036. \tag{4.5}$$

The robust optimal $H_\infty$ memoryless state feedback controller is given as

$$\boldsymbol{K} = [\text{-3.5435} \quad \text{-16.1184}]. \tag{4.6}$$

## 5. CONCLUSION

This paper has presented a solution to the problem of robust optimal $H_\infty$ control for uncertain 2-D discrete state-delayed systems described by the GM via memory state feedback. A sufficient condition for uncertain 2-D discrete state-delayed systems to have a specified $H_\infty$ noise attenuation is proposed in terms of a certain LMI. By solving a convex optimization problem the desired robust optimal $H_\infty$ controller is obtained. Finally, an example is given to illustrate the effectiveness of the proposed approach. A comparison of our result with previously reported result *i.e.* state feedback without memory (Singh and Dhawan, 2016) is also made in example 1 and it is found that our proposed method provides improved performance than memoryless state feedback controller.

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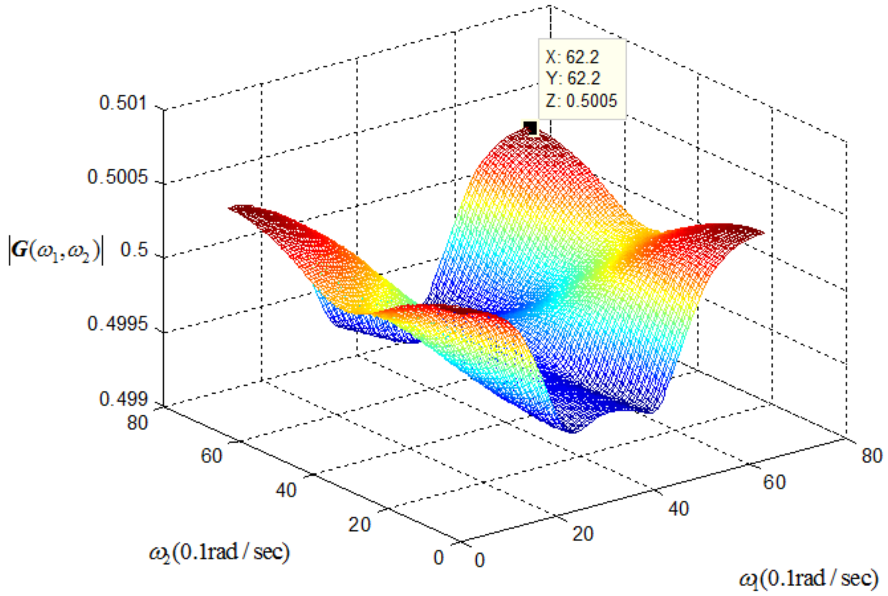


**Figure 1: The frequency response**